

# The Harmonic Oscillator with Dissipation within the Theory of Open Quantum Systems

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*Abstract.* Time evolution of the expectation values of various dynamical operators of the harmonic oscillator with dissipation is analitically obtained within the framework of the Lindblad theory for open quantum systems. We deduce the density matrix of the damped harmonic oscillator from the solution of the Fokker-Planck equation for the coherent state representation, obtained from the master equation for the density operator. The Fokker-Planck equation for the Wigner distribution function, subject to either the Gaussian type or the  $\delta$ -function type of initial conditions, is also solved by using the Wang-Uhlenbeck method. The obtained Wigner functions are two-dimensional Gaussians with different widths.

## 1 Introduction

In the last two decades, the problem of dissipation in quantum mechanics, i.e. the consistent description of open quantum systems, was investigated by various authors [1-6]. Because dissipative processes imply irreversibility and, therefore, a preferred direction in time, it is generally thought that quantum dynamical semigroups are the basic tools to introduce dissipation in quantum mechanics. The most general form of the generators of such semigroups was given by Lindblad [7-9]. This formalism has been studied for the case of damped harmonic oscillators [8, 10, 11] and applied to various physical phenomena, for instance, the damping of collective modes in deep inelastic collisions in nuclear physics [12, 13] and the interaction of a two-level atom with the electromagnetic field [14]. Recently [15], a family

of master equations, constructed in the form of Lindblad generators, was proposed for local ohmic quantum dissipation.

This paper, dealing with the damping of the harmonic oscillator within the Lindblad theory for open quantum systems, is concerned with the time evolution of various dynamical operators involved in the master and Fokker-Planck equations, in particular with the time development of the density matrix. In [16] the Lindblad master equation was transformed into Fokker-Planck equations for quasiprobability distributions and a comparative study was made for the Glauber  $P$ , antinormal ordering  $Q$  and Wigner  $W$  representations. In [17] the density matrix of the damped harmonic oscillator was represented by a generating function. We shall explore the physical aspects of the Fokker-Planck equation which is the  $c$ -number equivalent equation to the master equation for the density operator. Generally the master equation gains considerably in clarity if it is represented in terms of the Wigner distribution function which satisfies the Fokker-Planck equation. It is worth mentioning that these master and Fokker-Planck equations agree in form with the corresponding equations formulated in quantum optics [18-25].

The content of the paper is arranged as follows. In Sec.2 we review the derivation of the master equation of the harmonic oscillator. In order to get an insight into physical meanings of this equation, we first split it up into several equations satisfied by the expectation values of dynamical operators involved in the master equation. These equations are then solved analytically. In Sec.3 we transform the master equation into the Fokker-Planck equation by means of the well-known methods [4,26-30]. We extract the density matrix with the help of the solution of the Fokker-Planck equation for the coherent state representation. Then the Fokker-Planck equation for the Wigner distribution, subject to either the Gaussian type or the  $\delta$ -function type of initial conditions, is solved by the Wang-Uhlenbeck method. Finally, conclusions are given in Sec.4.

## 2 Master equation for the damped harmonic oscillator

The rigorous formulation for introducing the dissipation into a quantum mechanical system is that of quantum dynamical semigroups [2, 3, 7]. According to the axiomatic theory of Lindblad [7, 9], the usual von Neumann-Liouville equation ruling the time evolution of closed quantum systems is replaced in the case of open systems by the following equation for the density operator  $\rho$ :

$$\frac{d\Phi_t(\rho)}{dt} = L(\Phi_t(\rho)). \quad (2.1)$$

Here,  $\Phi_t$  denotes the dynamical semigroup describing the irreversible time evolution of the open system in the Schrödinger representation and  $L$  the infinitesimal generator of the dynamical semigroup  $\Phi_t$ . Using the structural theorem of Lindblad [7] which gives the most general form of the bounded, completely dissipative Liouville operator  $L$ , we obtain the explicit form of the most general time-homogeneous quantum mechanical Markovian master

equation:

$$\frac{d\rho(t)}{dt} = L(\rho(t)) = -\frac{i}{\hbar}[H, \rho(t)] + \frac{1}{2\hbar} \sum_j ([V_j \rho(t), V_j^\dagger] + [V_j, \rho(t) V_j^\dagger]). \quad (2.2)$$

Here  $H$  is the Hamiltonian of the system. The operators  $V_j$  and  $V_j^\dagger$  are bounded operators on the Hilbert space of the Hamiltonian.

We should like to mention that the Markovian master equations found in the literature are of this form after some rearrangement of terms, even for unbounded Liouville operators. In this connection we assume that the general form of the master equation given by (2.2) is also valid for unbounded Liouville operators.

In this paper we impose a simple condition to the operators  $H, V_j, V_j^\dagger$  that they are functions of the basic observables  $q$  and  $p$  of the one-dimensional quantum mechanical system of such kind that the obtained model is exactly solvable. This condition implies [8] that  $V_j$  are at most first degree polynomials in  $p$  and  $q$  and  $H$  is at most a second degree polynomial in  $p$  and  $q$ . Because in the linear space of the first degree polynomials in  $p$  and  $q$  the operators  $p$  and  $q$  give a basis, there exist only two  $\mathbf{C}$ -linear independent operators  $V_1, V_2$  which can be written in the form

$$V_j = a_j p + b_j q, \quad j = 1, 2, \quad (2.3)$$

with  $a_j, b_j$  complex numbers [8]. The constant term is omitted because its contribution to the generator  $L$  is equivalent to terms in  $H$  linear in  $p$  and  $q$  which for simplicity are assumed to be zero. Then the harmonic oscillator Hamiltonian  $H$  is chosen of the form

$$H = H_0 + \frac{\mu}{2}(pq + qp), \quad H_0 = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}q^2. \quad (2.4)$$

With these choices and introducing the annihilation and creation operators via the relations

$$q = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a), \quad p = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a), \quad (2.5)$$

we have  $H_0 = \hbar\omega(a^\dagger a + 1/2)$  and the Markovian master equation takes the form

$$\begin{aligned} \frac{d\rho}{dt} = & \frac{1}{2}(D_1 + \mu)(a^\dagger a^\dagger \rho - a^\dagger \rho a^\dagger) + \frac{1}{2}(D_1 - \mu)(\rho a^\dagger a^\dagger - a^\dagger \rho a^\dagger) \\ & + \frac{1}{2}(D_2 + \lambda + i\omega)(a \rho a^\dagger - a^\dagger a \rho) + \frac{1}{2}(D_2 - \lambda - i\omega)(a^\dagger \rho a - \rho a a^\dagger) + \text{H.c.}, \end{aligned} \quad (2.6)$$

where

$$D_1 \equiv \frac{1}{\hbar}(m\omega D_{qq} - \frac{D_{pp}}{m\omega} + 2iD_{pq}), \quad D_2 \equiv \frac{1}{\hbar}(m\omega D_{qq} + \frac{D_{pp}}{m\omega}). \quad (2.7)$$

Here we used the notations:

$$D_{qq} = \frac{\hbar}{2} \sum_{j=1,2} |a_j|^2, \quad D_{pp} = \frac{\hbar}{2} \sum_{j=1,2} |b_j|^2, \quad D_{pq} = D_{qp} = -\frac{\hbar}{2} \text{Re} \sum_{j=1,2} a_j^* b_j, \quad \lambda = -\text{Im} \sum_{j=1,2} a_j^* b_j, \quad (2.8)$$

where  $D_{pp}$ ,  $D_{qq}$  and  $D_{pq}$  are the diffusion coefficients and  $\lambda$  the friction constant. They satisfy the following fundamental constraints [10]:

$$\text{i) } D_{pp} > 0, \quad \text{ii) } D_{qq} > 0, \quad \text{iii) } D_{pp}D_{qq} - D_{pq}^2 \geq \lambda^2 \hbar^2 / 4. \quad (2.9)$$

In the particular case when the asymptotic state is a Gibbs state

$$\rho_G(\infty) = e^{-\frac{H_0}{kT}} / \text{Tr} e^{-\frac{H_0}{kT}}, \quad (2.10)$$

these coefficients reduce to

$$D_{pp} = \frac{\lambda + \mu}{2} \hbar m \omega \coth \frac{\hbar \omega}{2kT}, \quad D_{qq} = \frac{\lambda - \mu}{2} \frac{\hbar}{m \omega} \coth \frac{\hbar \omega}{2kT}, \quad D_{pq} = 0, \quad (2.11)$$

where  $T$  is the temperature of the thermal bath.

In the literature, master equations of the type (2.6) are encountered in concrete theoretical models for the description of different physical phenomena in quantum optics [18-24], in treatments of the damping of collective modes in deep inelastic collisions of heavy ions [31-34] or in the quantum mechanical description of the dissipation for the one-dimensional harmonic oscillator [4, 6, 28, 29]. A classification of these equations, whether they satisfy or not the fundamental constraints (2.9), was given in [17].

The meaning of the master equation becomes clear when we transform it into equations satisfied by various expectation values of operators involved in the master equation,  $\langle A \rangle = \text{Tr}[\rho(t)A]$ , where  $A$  is an operator composed of the creation and annihilation operators. Multiplying both sides of (2.6) by  $a$  and taking throughout the trace, we get

$$\frac{d}{dt} \langle a \rangle = -(\lambda + i\omega) \langle a \rangle + \mu \langle a^\dagger \rangle. \quad (2.12)$$

Similarly, the equation for  $\langle a^\dagger \rangle$  is given by

$$\frac{d}{dt} \langle a^\dagger \rangle = -(\lambda - i\omega) \langle a^\dagger \rangle + \mu \langle a \rangle. \quad (2.13)$$

In the absence of the second term on the right-hand side of (2.12) and (2.13), the two equations represent independently a simple equation of oscillation with damping. By coupling (2.12) to (2.13) we get a second order differential equation for either  $\langle a \rangle$  or  $\langle a^\dagger \rangle$ . For example, we obtain:

$$\frac{d^2}{dt^2} \langle a \rangle + 2\lambda \frac{d}{dt} \langle a \rangle + (\lambda^2 + \omega^2 - \mu^2) \langle a \rangle = 0, \quad (2.14)$$

which is the equation of motion for Brownian motion of a classical oscillator, but without the term corresponding to random process. Because of the vanishing terms on the right-hand side, we may equally state that (2.14) is the equation of motion with zero expectation value of the random process. In the study of the Brownian motion of a classical oscillator, one replaces the second-order differential equation of motion of the type (2.14) by two equivalent first-order differential equations [35] which are precisely the Langevin equations. Accordingly,

we may say that (2.12) and (2.13) are the Langevin equations corresponding to (2.14), but without the random process term. The integration of (2.14) is straightforward. There are two cases: a)  $\mu > \omega$  (overdamped) and b)  $\mu < \omega$  (underdamped). In the case a) with the notation  $\nu^2 \equiv \mu^2 - \omega^2$ , we obtain:

$$\langle a(t) \rangle = e^{-\lambda t} [\langle a(0) \rangle (\cosh \nu t - i \frac{\omega}{\nu} \sinh \nu t) + \frac{\mu}{\nu} \langle a^\dagger(0) \rangle \sinh \nu t]. \quad (2.15a)$$

In the case b) with the notation  $\Omega^2 \equiv \omega^2 - \mu^2$ , we obtain:

$$\langle a(t) \rangle = e^{-\lambda t} [\langle a(0) \rangle (\cos \Omega t - i \frac{\omega}{\Omega} \sin \Omega t) + \frac{\mu}{\Omega} \langle a^\dagger(0) \rangle \sin \Omega t]. \quad (2.15b)$$

The expression for  $\langle a^\dagger(t) \rangle$  can be obtained simply by taking the complex conjugate of the right-hand side of (2.15).

For the computation of quantal fluctuations of the coordinate and momentum of the harmonic oscillator, we need the expectation values of quadratic operators, such as  $a^{+2}$ ,  $a^2$  or  $a^\dagger a$ . The dynamical behaviour of these operators can be well surveyed by deriving the equations satisfied by their expectation values. By following the same procedure as before, employed in the derivation of (2.12), we find:

$$\frac{d}{dt} \langle a^2 \rangle + 2(\lambda + i\omega) \langle a^2 \rangle = 2\mu \langle a^\dagger a \rangle + D_1 + \mu, \quad (2.16)$$

$$\frac{d}{dt} \langle a^{+2} \rangle + 2(\lambda - i\omega) \langle a^{+2} \rangle = 2\mu \langle a^\dagger a \rangle + D_1^* + \mu, \quad (2.17)$$

$$\frac{d}{dt} \langle a^\dagger a \rangle + 2\lambda \langle a^\dagger a \rangle = \mu(\langle a^{+2} \rangle + \langle a^2 \rangle) + D_2 - \lambda. \quad (2.18)$$

The solutions of these equations are readily obtained by transforming them into two differential equations satisfied by the sum and the difference of two quadratic operators,  $\langle a^2 \rangle$  and  $\langle a^{+2} \rangle$ . We obtain:

$$\begin{aligned} \frac{1}{4} \frac{d^2}{dt^2} (\langle a^2 \rangle + \langle a^{+2} \rangle) + \lambda \frac{d}{dt} (\langle a^2 \rangle + \langle a^{+2} \rangle) + (\lambda^2 + \omega^2 - \mu^2) (\langle a^2 \rangle + \langle a^{+2} \rangle) \\ = \frac{1}{\hbar} [(\lambda + \mu)m\omega D_{qq} - (\lambda - \mu) \frac{D_{pp}}{m\omega} + 2\omega D_{pq}] \equiv D, \end{aligned} \quad (2.19)$$

$$\frac{1}{2} \frac{d}{dt} (\langle a^2 \rangle - \langle a^{+2} \rangle) + \lambda (\langle a^2 \rangle - \langle a^{+2} \rangle) + i\omega (\langle a^2 \rangle + \langle a^{+2} \rangle) = 2i \frac{D_{pq}}{\hbar}. \quad (2.20)$$

The solution of (2.19) is straightforward and with the help of which both equations (2.18) and (2.20) can be immediately solved. We find:

$$\langle a^2 \rangle = e^{-2\lambda t} [(1 - i \frac{\omega}{\nu}) C_1 e^{2\nu t} + (1 + i \frac{\omega}{\nu}) C_2 e^{-2\nu t} - i \frac{\mu}{\omega} C_3] + \frac{D(\lambda - i\omega)}{2\lambda(\lambda^2 - \nu^2)} + i \frac{D_{pq}}{\hbar\lambda}, \quad (2.21a)$$

$$\langle a^\dagger a \rangle = e^{-2\lambda t} [\frac{\mu}{\nu} (C_1 e^{2\nu t} - C_2 e^{-2\nu t}) + C_3] + \frac{1}{2\lambda} (\frac{D\mu}{\lambda^2 - \nu^2} + D_2 - \lambda), \quad (2.22a)$$

for the overdamped case  $\mu > \omega$  and

$$\langle a^2 \rangle = e^{-2\lambda t} \left[ (C_1 + iC_2 \frac{\omega}{\Omega}) \cos 2\Omega t + (C_2 - iC_1 \frac{\omega}{\Omega}) \sin 2\Omega t - i\frac{\mu}{\omega} C_3 \right] + \frac{D(\lambda - i\omega)}{2\lambda(\lambda^2 + \Omega^2)} + i\frac{D_{pq}}{\hbar\lambda}, \quad (2.21b)$$

$$\langle a^\dagger a \rangle = e^{-2\lambda t} \left[ \frac{\mu}{\Omega} (C_1 \sin 2\Omega t - C_2 \cos 2\Omega t) + C_3 \right] + \frac{1}{2\lambda} \left( \frac{D\mu}{\lambda^2 + \Omega^2} + D_2 - \lambda \right), \quad (2.22b)$$

for the underdamped case  $\omega > \mu$ . The expression for  $\langle a^{+2} \rangle$  can be obtained by taking the complex conjugates of (2.21a) and (2.21b). Here,  $C_1, C_2, C_3$  are the integral constants depending on the initial expectation values of the operators under consideration. In particular, if  $D_{qq} = D_{pq} = 0$  and  $\mu = \lambda$  we obtain for the underdamped case  $\omega > \mu$  the equations written by Jang [31] for the model on nuclear dynamics based on the second RPA at finite temperature. For time  $t \rightarrow \infty$ , we see from (2.22) that

$$\langle a^\dagger a \rangle = \frac{1}{2\lambda} \left( \frac{D\mu}{\lambda^2 + \omega^2 - \mu^2} + D_2 - \lambda \right). \quad (2.23)$$

In the particular case when the asymptotic state is a Gibbs state (2.10), we get

$$\langle a^\dagger a \rangle = \frac{1}{2} \left( \coth \frac{\hbar\omega}{2kT} - 1 \right) = \left( \exp \frac{\hbar\omega}{kT} - 1 \right)^{-1} \equiv \langle n \rangle, \quad (2.24)$$

which is the Bose distribution. This means that the expectation value of the number operator goes to the average thermal-phonon number at infinity of time. From the identity

$$\langle a^\dagger a \rangle = \sum_{m=0}^{\infty} m \langle m | \rho(t) | m \rangle \quad (2.25)$$

it follows

$$\langle m | \rho(\infty) | m \rangle = \frac{\langle n \rangle^m}{(1 + \langle n \rangle)^{m+1}}. \quad (2.26)$$

In deriving this formula, we have made use of the identity  $\sum_{m=0}^{\infty} mx^m = x/(1-x)^2$ . The expression (2.26) shows that in the considered particular case the density matrix reaches its thermal equilibrium – the Bose-Einstein distribution, whatever the initial distribution of the density matrix may be. When the initial density matrix  $\langle m | \rho(0) | m \rangle$  is represented by a distribution of the form  $N^m/(1+N)^{m+1}$ , where  $N$  stands for the average phonon number, the relation (2.25) implies that  $\langle a^\dagger a \rangle = N$ . When the initial density matrix is characterized by a distribution of the form

$$\frac{1}{m!} N^m e^{-N}, \quad (2.27)$$

(2.25) implies that  $\langle a^\dagger a \rangle$  becomes also  $N$ . Eq. (2.27) is nothing but a Poisson distribution. If the initial density matrix is represented by a Kronecker delta  $\delta_{ms}$ , we see from (2.25) that  $\langle a^\dagger a \rangle = s$ , which corresponds to the initial  $s$ -phonon state.

The physical observables of the harmonic oscillator can be obtained from the expectation values of polynomials of the annihilation and creation operators. So, for the position and momentum operators  $q$  and  $p$  via the relations (2.5), we can evaluate either the second moments or variances (fluctuations), by making use of the results (2.15), (2.21), (2.22).

### 3 Fokker-Planck equations

One useful way to study the consequences of the master equation (2.6) for the density operator of the one-dimensional damped harmonic oscillator is to transform it into more familiar forms, such as the equations for the  $c$ -number quasiprobability distributions Glauber  $P$ , anti-normal ordering  $Q$  and Wigner  $W$  associated with the density operator [16]. In this case the resulting differential equations of the Fokker-Planck type for the distribution functions can be solved by standard methods [26-29] employed in quantum optics and observables directly calculated as correlations of these distribution functions.

#### 3.1 Calculation of the density matrix from the Fokker-Planck equation

The Fokker-Planck equation, obtained from the master equation and satisfied by the Wigner distribution function  $W(\alpha, \alpha^*, t)$ , where  $\alpha$  is a complex variable, has the form [16]:

$$\begin{aligned} \frac{\partial W(\alpha, \alpha^*, t)}{\partial t} = & -\left\{ \frac{\partial}{\partial \alpha} [-(\lambda + i\omega)\alpha + \mu\alpha^*] + \frac{\partial}{\partial \alpha^*} [-(\lambda - i\omega)\alpha^* + \mu\alpha] \right\} W(\alpha, \alpha^*, t) \\ & + \frac{1}{2} (D_1 \frac{\partial^2}{\partial \alpha^2} + D_1^* \frac{\partial^2}{\partial \alpha^{*2}} + 2D_2 \frac{\partial^2}{\partial \alpha \partial \alpha^*}) W(\alpha, \alpha^*, t). \end{aligned} \quad (3.1)$$

When we substitute the  $P$  representation function  $P(\alpha, \alpha^*, t)$  for  $W(\alpha, \alpha^*, t)$  and the coefficients  $D_1 + \mu$  for  $D_1$ ,  $D_2 - \lambda$  for  $D_2$  in the above equation, we get the Fokker-Planck equation for the coherent representation [16].

The Fokker-Planck equation for the  $P$  representation, subject to the initial condition

$$P(\alpha, \alpha^*, 0) = \delta(\alpha - \alpha_0) \delta(\alpha^* - \alpha_0^*), \quad (3.2)$$

where  $\alpha_0$  is the initial value of  $\alpha$  can be solved [30, 36, 37] and the solution (Green function) for the  $P$  representation is found to be

$$P(\alpha, \alpha^*, t) = \frac{2}{\pi \sqrt{\det \sigma(t)}} \exp \left\{ -\frac{1}{\det \sigma(t)} [\sigma_{22}(\alpha - \bar{\alpha}_0)^2 + \sigma_{11}(\alpha^* - \bar{\alpha}_0^*)^2 - 2\sigma_{12}|\alpha - \bar{\alpha}_0|^2] \right\}, \quad (3.3)$$

where

$$\sigma_{ij}(t) = \sum_{s,r=1,2} [\delta_{is} \delta_{jr} - b_{is}(t) b_{jr}(t)] \sigma_{sr}(\infty). \quad (3.4)$$

The function  $\bar{\alpha}_0$  and its complex conjugate, which are still functions of time, are given by

$$\bar{\alpha}_0 = b_{11}(t) \alpha_0 + b_{12}(t) \alpha_0^*. \quad (3.5)$$

The functions  $b_{ij}$  obey the equations

$$\dot{b}_{is} = \sum_{j=1,2} c_{ij} b_{js} \quad (3.6)$$

with the initial conditions  $b_{js}(0) = \delta_{js}$  and  $\sigma(\infty)$  is determined by

$$C\sigma(\infty) + \sigma(\infty)C^T = Q^P, \quad (3.7)$$

where

$$C = \begin{pmatrix} \lambda + i\omega & -\mu \\ -\mu & \lambda - i\omega \end{pmatrix}, \quad Q^P = \begin{pmatrix} D_1 + \mu & D_2 - \lambda \\ D_2 - \lambda & D_1^* + \mu \end{pmatrix}. \quad (3.8)$$

We get

$$b_{11} = b_{22}^* = e^{-\lambda t}(\cos \Omega t - i\frac{\omega}{\Omega} \sin \Omega t), \quad b_{12} = b_{21} = \frac{\mu}{\Omega} e^{-\lambda t} \sin \Omega t, \quad (3.9)$$

with  $\Omega^2 \equiv \omega^2 - \mu^2$ . While the functions  $\sigma_{11}, \sigma_{22}$  and  $\bar{\alpha}_0$  are complex with  $\sigma_{11} = \sigma_{22}^*$ , the functions  $\det \sigma(t)$  and  $\sigma_{12}$  are real.

The solution of the Fokker-Planck equation has been written down providing the diffusion matrix  $Q^P$  is positive definite. However, the diffusion matrix in the Glauber  $P$  representation is not, in general, positive definite. If the  $P$  distribution does not exist as a well-behaved function, the so-called generalized  $P$  distributions can be taken that are well-behaved, normal ordering functions [38].

In the coherent representation [39, 40] the density operator  $\rho(t)$  is expressed by

$$\rho(t) = \int P(\alpha, \alpha^*, t) |\alpha\rangle \langle \alpha| d^2\alpha, \quad (3.10)$$

where  $d^2\alpha = d(\text{Re}\alpha)d(\text{Im}\alpha)$  and  $|\alpha\rangle$  is the coherent state. The matrix element of  $\rho(t)$  in the  $n$  quantum number representation is obtained by multiplying (3.10) on the left by  $\langle m|$  and on the right by  $|n\rangle$ . By making use of the well-known relation

$$|\alpha\rangle = \exp(-\frac{1}{2}|\alpha|^2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad (3.11)$$

we get

$$\langle m|\rho(t)|n\rangle = \frac{1}{\sqrt{m!n!}} \int \alpha^n \alpha^{*m} P(\alpha, \alpha^*, t) \exp(-|\alpha|^2) d^2\alpha. \quad (3.12)$$

Upon introducing the explicit form (3.3) for  $P(\alpha, \alpha^*, t)$  into (3.12), we obtain the desired density matrix for the initial coherent state. However, due to the powers of complex variables  $\alpha$  and  $\alpha^*$  in the integrand, the practical evaluation of the integral in (3.12) is not an easy task. Instead, we use the method of generating function [31] which allows us to transform (3.12) into a multiple-differential form. When we define a generating function  $F(x, y, t)$  by the integral

$$F(x, y, t) = \int P(\alpha, \alpha^*, t) \exp(-|\alpha|^2 + x\alpha + y\alpha^*) d^2\alpha, \quad (3.13)$$

we see that the density matrix is related to the generating function by

$$\langle m|\rho(t)|n\rangle = \frac{1}{\sqrt{m!n!}} \left(\frac{\partial}{\partial x}\right)^m \left(\frac{\partial}{\partial y}\right)^n F(x, y, t) |_{x=y=0}. \quad (3.14)$$



Since the  $P$  representation is in a Gaussian form, the right-hand side of (3.13) can be evaluated analitically by making use of the identity

$$\int \exp(-a|z|^2 + bz + cz^* + ez^2 + fz^{*2})d^2z = \frac{\pi}{\sqrt{a^2 - 4ef}} \exp \frac{abc + b^2f + c^2e}{a^2 - 4ef}, \quad (3.15)$$

which is convergent for  $\text{Re } a > |e^* + f|$ , while  $b, c$  may be arbitrary. We find:

$$F = \frac{2}{\sqrt{|A|}} \exp\{xy - \frac{1}{A}[\sigma_{11}(x - \bar{\alpha}_0^*)^2 + \sigma_{22}(y - \bar{\alpha}_0)^2 - 2(\sigma_{12} + 2)(x - \bar{\alpha}_0^*)(y - \bar{\alpha}_0)]\}, \quad (3.16)$$

where  $A \equiv d - 4(\sigma_{12} + 1)$ ,  $d \equiv \det \sigma = \sigma_{11}\sigma_{22} - \sigma_{12}^2$ . A formula for the density matrix can be written down by applying the relation (3.14) to the generating function (3.16). We get

$$\begin{aligned} \langle m|\rho(t)|n \rangle &= 2 \frac{\sqrt{m!n!}}{\sqrt{|A|}} \exp\left\{[-\frac{1}{A}[\sigma_{22}\bar{\alpha}_0^2 + \sigma_{11}\bar{\alpha}_0^{*2} - 2(\sigma_{12} + 2)|\bar{\alpha}_0|^2]]\right\} \\ &\times \sum_{n_1, n_2, n_3=0} \frac{(-1)^{n_1+n_2} 2^{m+n-2(n_1+n_2+n_3)}}{n_1!n_2!n_3!(m-2n_1-n_3)!(n-2n_2-n_3)!} E, \end{aligned} \quad (3.17)$$

where

$$E = \frac{\sigma_{11}^{n_1} \sigma_{22}^{n_2} (d - 2\sigma_{12})^{n_3} [\sigma_{11}\bar{\alpha}_0^* - (\sigma_{12} + 2)\bar{\alpha}_0]^{m-2n_1-n_3} [\sigma_{22}\bar{\alpha}_0 - 2(\sigma_{12} + 2)\bar{\alpha}_0^*]^{n-2n_2-n_3}}{A^{m+n-(n_1+n_2+n_3)}}.$$

The expression (3.17) is the density matrix corresponding to the initial coherent state. At time  $t = 0$ , the functions  $\sigma_{11}$ ,  $\sigma_{22}$  and  $\sigma_{12}$  vanish and  $\bar{\alpha}_0$  goes to  $\alpha_0$ . In this case the density matrix reduces to

$$\langle m|\rho(0)|n \rangle = \frac{1}{\sqrt{m!n!}} \alpha_0^n \alpha_0^{*m} \exp(-|\alpha_0|^2), \quad (3.18)$$

which is the initial Glauber packet. For the diagonal case the initial density matrix becomes the Poisson distribution. At infinity of time, the density matrix (3.17) goes to the Bose-Einstein distribution

$$\langle m|\rho(\infty)|n \rangle = \frac{\langle n \rangle^m}{(1 + \langle n \rangle)^{m+1}} \delta_{mn}. \quad (3.19)$$

### 3.2 Wigner distribution function

The Fokker-Planck equation (3.1) can also be written in terms of real coordinates  $x_1$  and  $x_2$  (or the averaged position and momentum coordinates of the harmonic oscillator) defined by  $\alpha = x_1 + ix_2$ ,  $\alpha^* = x_1 - ix_2$ , as follows:

$$\frac{\partial W}{\partial t} = \sum_{i,j=1,2} A_{ij} \frac{\partial}{\partial x_i} (x_j W) + \frac{1}{2} \sum_{i,j=1,2} Q_{ij}^W \frac{\partial^2}{\partial x_i \partial x_j} W, \quad (3.20)$$

where

$$A = \begin{pmatrix} \lambda - \mu & -\omega \\ \omega & \lambda + \mu \end{pmatrix}, \quad Q^W = \frac{1}{\hbar} \begin{pmatrix} m\omega D_{qq} & D_{pq} \\ D_{pq} & D_{pp}/m\omega \end{pmatrix}. \quad (3.21)$$

Since the drift coefficients are linear in the variables  $x_1$  and  $x_2$  and the diffusion coefficients are constant with respect to  $x_1$  and  $x_2$ , (3.20) describes an Ornstein-Uhlenbeck process [35, 41]. Following the method developed by Wang and Uhlenbeck [35], we shall solve this Fokker-Planck equation, subject to either the wave-packet type or the  $\delta$ -function type of initial conditions. By changing the variables  $x_1$  and  $x_2$  of (3.20) via the relations

$$z_1 = ax_1 + bx_2, \quad z_2 = cx_1 + dx_2, \quad (3.22)$$

(3.20) is transformed into the standard partial differential equation [35] expressed as

$$\frac{\partial W(z_1, z_2, t)}{\partial t} = \left( - \sum_{i=1,2} \nu_i \frac{\partial}{\partial z_i} z_i + \frac{1}{2} \sum_{i,j=1,2} D_{ij} \frac{\partial^2}{\partial z_i \partial z_j} \right) W(z_1, z_2, t). \quad (3.23)$$

In deriving this equation we have put  $a = c^* = (\mu - i\Omega)/\omega$ ,  $b = d = 1$ ,  $\nu_1 = \nu_2^* \equiv -\lambda - i\Omega$  and

$$\begin{aligned} D_{11} = D_{22}^* &\equiv \frac{1}{\hbar\omega} [(\mu - i\Omega)^2 m D_{qq} + 2(\mu - i\Omega) D_{pq} + \frac{D_{pp}}{m}], \\ D_{12} = D_{21} &\equiv \frac{1}{\hbar} (m\omega D_{qq} + \frac{2\mu}{\omega} D_{pq} + \frac{D_{pp}}{m\omega}). \end{aligned} \quad (3.24)$$

1) When the Fokker-Planck equation for the coherent state representation is subject to the initial condition  $\delta(\alpha - \alpha_0)\delta(\alpha^* - \alpha_0^*)$ , then the use of the relation between the Wigner distribution function and  $P$  representation

$$W(\alpha, \alpha^*, t) = \frac{2}{\pi} \int P(\beta, \beta^*, t) \exp(-2|\alpha - \beta|^2) d^2\beta \quad (3.25)$$

leads to a Gaussian form for the initial Wigner function. If this Wigner function is expressed in terms of  $x_{10}$  and  $x_{20}$  – the initial values of  $x_1$  and  $x_2$  at  $t = 0$ , respectively, then we get the expression which corresponds to the initial condition of a wave packet:

$$W_w(x_1, x_2, 0) = \frac{1}{2\hbar} W_w(\alpha, \alpha^*, 0) = \frac{1}{\pi\hbar} \exp(-2|\alpha - \alpha_0|^2) = \frac{1}{\pi\hbar} \exp\{-2[(x_1 - x_{10})^2 + (x_2 - x_{20})^2]\}. \quad (3.26)$$

Accordingly, we now look for the solution of the Fokker-Planck equation (3.20) subject to (3.26). By changing the variables  $x_1$  and  $x_2$  into  $z_1$  and  $z_2$ , ( $z_1 = z_2^* \equiv z$ ), this initial condition is seen to be transformed into

$$W_w(z, z^*, 0) = \frac{1}{\pi\hbar} \exp\left\{ \frac{2\omega^2}{\Omega^2} [q(z - z_0)^2 + q^*(z^* - z_0^*)^2 - |z - z_0|^2] \right\}, \quad (3.27)$$

where  $z_0$  is the initial value of  $z$  and  $q = \mu(\mu + i\Omega)/2\omega^2$ . The solution of (3.23) subject to the initial condition (3.27) is found to be

$$W_w(z, z^*, t) = \frac{\Omega}{\pi\hbar\omega\sqrt{|B_w|}} \exp\left\{ -\frac{1}{2B_w} [g_2(z - z_0 e^{\nu_1 t})^2 + g_1(z^* - z_0^* e^{\nu_2 t})^2 - g_3|z - z_0 e^{\nu_1 t}|^2] \right\}, \quad (3.28)$$

where

$$B_w = g_1 g_2 - \frac{1}{4} g_3^2, g_1 = g_2^* = q^* e^{2\nu_1 t} + \frac{D_{11}}{2\nu_1} (e^{2\nu_1 t} - 1), g_3 = e^{-2\lambda t} + \frac{D_{12}}{\lambda} (1 - e^{-2\lambda t}). \quad (3.29)$$

In terms of real variables  $x_1$  and  $x_2$  we have:

$$W_w(x_1, x_2, t) = \frac{\Omega}{\pi \hbar \omega \sqrt{|B_w|}} \exp\left\{-\frac{1}{2B_w} [\phi_w(x_1 - \bar{x}_1)^2 + \psi_w(x_2 - \bar{x}_2)^2 + \chi_w(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)]\right\}, \quad (3.30)$$

where

$$\phi_w = g_1 a^{*2} + g_2 a^2 - g_3, \quad \psi_w = g_1 + g_2 - g_3, \quad \chi_w = 2(g_1 a^* + g_2 a) - g_3(a + a^*). \quad (3.31)$$

The functions  $\bar{x}_1$  and  $\bar{x}_2$ , which are also oscillating functions, are given by

$$\begin{aligned} \bar{x}_1 &= e^{-\lambda t} \left[ x_{10} (\cos \Omega t + \frac{\mu}{\Omega} \sin \Omega t) + x_{20} \frac{\omega}{\Omega} \sin \Omega t \right], \\ \bar{x}_2 &= e^{-\lambda t} \left[ x_{20} (\cos \Omega t - \frac{\mu}{\Omega} \sin \Omega t) - x_{10} \frac{\omega}{\Omega} \sin \Omega t \right]. \end{aligned} \quad (3.32)$$

2) If the Fokker-Planck equation (3.23) is subject to the  $\delta$ -function type of initial condition, the Wigner distribution function is given by

$$W(z, z^*, t) = \frac{\Omega}{\pi \hbar \omega \sqrt{|B|}} \exp\left\{-\frac{1}{B} [f_2(z - z_0 e^{\nu_1 t})^2 + f_1(z^* - z_0^* e^{\nu_2 t})^2 - 2f_3|z - z_0 e^{\nu_1 t}|^2]\right\}, \quad (3.33)$$

where

$$B = f_1 f_2 - f_3^2, \quad f_1 = f_2^* = \frac{D_{11}}{\nu_1} (e^{2\nu_1 t} - 1), \quad f_3 = \frac{D_{12}}{\lambda} (1 - e^{-2\lambda t}). \quad (3.34)$$

In terms of real variables  $x_1$  and  $x_2$  we have:

$$W(x_1, x_2, t) = \frac{\Omega}{\pi \hbar \omega \sqrt{|B|}} \exp\left\{-\frac{1}{B} [\phi_d(x_1 - \bar{x}_1)^2 + \psi_d(x_2 - \bar{x}_2)^2 + \chi_d(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)]\right\}, \quad (3.35)$$

where

$$\phi_d = f_1 a^{*2} + f_2 a^2 - 2f_3, \quad \psi_d = f_1 + f_2 - 2f_3, \quad \chi_d = 2[f_1 a^* + f_2 a - f_3(a + a^*)]. \quad (3.36)$$

So, one gets a 2-dimensional Gaussian distribution with the average values  $\bar{x}_1$  and  $\bar{x}_2$  and the variances  $\phi_d, \psi_d$  and  $\chi_d$ .

When time  $t \rightarrow \infty$ ,  $\bar{x}_1$  and  $\bar{x}_2$  vanish and we obtain the steady state solution:

$$W(x_1, x_2) = \frac{1}{2\pi \sqrt{\det \sigma^W(\infty)}} \exp\left[-\frac{1}{2} \sum_{i,j=1,2} (\sigma^W)_{ij}^{-1}(\infty) x_i x_j\right]. \quad (3.37)$$

The stationary covariance matrix  $\sigma^W(\infty)$  can be determined from the algebraic equation

$$A \sigma^W(\infty) + \sigma^W(\infty) A^T = Q^W. \quad (3.38)$$

We obtain:

$$\begin{aligned}
\sigma_{11}^W(\infty) &= \frac{(2\lambda(\lambda + \mu) + \omega^2)Q_{11}^W + \omega^2 Q_{22}^W + 2\omega(\lambda + \mu)Q_{12}^W}{4\lambda(\lambda^2 + \omega^2 - \mu^2)}, \\
\sigma_{22}^W(\infty) &= \frac{\omega^2 Q_{11}^W + (2\lambda(\lambda - \mu) + \omega^2)Q_{22}^W - 2\omega(\lambda - \mu)Q_{12}^W}{4\lambda(\lambda^2 + \omega^2 - \mu^2)}, \\
\sigma_{12}^W(\infty) &= \frac{-\omega(\lambda + \mu)Q_{11}^W + \omega(\lambda - \mu)Q_{22}^W + 2(\lambda^2 - \mu^2)Q_{12}^W}{4\lambda(\lambda^2 + \omega^2 - \mu^2)}.
\end{aligned} \tag{3.39}$$

## 4 Conclusions

Recently we assist to a revival of interest in quantum Brownian motion as a paradigm of quantum open systems. There are many motivations. The possibility of preparing systems in macroscopic quantum states led to the problems of dissipation in tunneling and of loss of quantum coherence (decoherence). These problems are intimately related to the issue of quantum-to-classical transition. All of them point the necessity of a better understanding of open quantum systems and all requires the extension of the model of quantum Brownian motion. The Lindblad theory provides a selfconsistent treatment of damping as a possible extension of quantum mechanics to open systems. In the present paper we have studied the one-dimensional harmonic oscillator with dissipation within the framework of this theory. We have carried out a calculation of the expectation values of various dynamical operators involved in the master equation, especially the first two moments and the density matrix. Generally, the time evolution of the density matrix as well as the expectation values of dynamical operators are characterized by complex functions with an oscillating element  $\exp(\pm i\sqrt{\omega^2 - \mu^2}t)$  multiplied by the damping factor  $\exp(-\lambda t)$ . We deduced the density matrix from the solution of the Fokker-Planck equation for the coherent state representation, obtained from the master equation for the density operator. For a thermal bath, when the asymptotic state is a Gibbs state, a Bose-Einstein distribution results as density matrix. The density matrix can be used in various physical applications where a Bosonic degree of freedom moving in a harmonic oscillator potential is damped. For example, one needs to determine nondiagonal transition elements of the density matrix, if an oscillator is perturbed by a weak electromagnetic field in addition to its coupling to a heat bath. From the master equation of the damped quantum oscillator we have also derived the corresponding Fokker-Planck equation in the Wigner  $W$  representation. The obtained equation describes an Ornstein-Uhlenbeck process. By using the Wang-Uhlenbeck method we have solved this equation for the Wigner function, subject to either the Gaussian type or the  $\delta$ -function type of initial conditions and showed that the Wigner functions are two-dimensional Gaussians with different widths. In a forthcoming paper, by using this Wigner function, it will be discussed the entropy of the damped harmonic oscillator.

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